Clifford Theory for Crossed Product Algebras

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1. Crossed Product Algebras

Definition 1.1. *Let G be a group acting on an algebra R over $\mathbb{C}$. The crossed product algebra is*

$$R \rtimes \mathbb{C}[G] = R \otimes \mathbb{C}[G]$$

with multiplication given by

$$(r \otimes g)(r' \otimes g') = r(g \cdot r') \otimes gg'$$[1]

Given $N \in R - \text{mod}$, $g \in G$, consider the new $R - \text{mod}$ ${}^gN = N$ as a set with twisted action

$$r \circ n = g^{-1}(r) \cdot n$$

Lemma 1.2. *${}^gN$ is simple $\iff$ N is simple.*

Proof. G acts by automorphisms on R so $M \subset N$ is a submodule $\iff$ ${}^gM \subset^g N$ is a submodule. ■

Thus G acts on the set of simple $R-$modules.

Definition 1.3. *Given N a simple $R - \text{mod}$, the inertia subgroup of N is*

$$I_N = \{g \in G | \, {}^gN \cong N\}$$

For $h \in I_N$, fix an isomorphism $\phi_h : N \to^{h^{-1}} N$, then

$$\phi_g \phi_h = \alpha_N(g, h)\phi_{gh}$$

where $\alpha_N(g, h) \in \mathbb{C}^\times$ by Schur's Lemma. Since ϕ_h is an intertwiner we have $\phi_h r = h(r)\phi_h$ and this will show that $\alpha_N(g, h) \in H^2(I_N, \mathbb{C}^\times)$.

Definition 1.4 (Twisted Group Algebra)**.** *Given $\alpha \in H^2(H, \mathbb{C}^\times)$, let $(\mathbb{C}H)_{\alpha^{-1}}$ be the algebra with underlying set $\mathbb{C}H$ with multiplication given by*

$$c_g c_h = \alpha(g, h)^{-1} c_{gh}$$

Let L^μ be a simple $(\mathbb{C}I_N)_{\alpha_N^{-1}} - \text{mod}$. Then there is an action of of $R \rtimes \mathbb{C}[I_N]$ on $N \otimes L^\mu$ given by

$$(r \otimes h)(n \otimes l) = r \cdot \phi_h(n) \otimes c_h \cdot l$$

(There's no typo, we have $\mathbb{C}[I_N]$ instead of $(\mathbb{C}I_N)_{\alpha_N^{-1}}$ because of double cancellation.)

[1] Equivalently, multiplication is given by concatenation using the commutation relation $gr = g(r)g$ to move all the G terms to the right.

Theorem 1 (Clifford Theory)
Let N be a simple $R-\text{mod}$, and L^μ a simple $(\mathbb{C}I_N)_{\alpha_N^{-1}}-\text{mod}$. Let

$$RG^{N,\mu} = \text{Ind}_{R\rtimes I_N}^{R\rtimes G}(N\otimes L^\mu)$$

(a) $RG^{N,\mu}$ is a simple $R\rtimes G$ module.

(b) Every simple $R\rtimes G$ module is obtained by this construction.

(c) $RG^{N_1,\mu}\cong RG^{N_2,\gamma} \iff N_1, N_2$ in same $G-$orbit and $L^\mu\cong L^\gamma$.

Proof. (a) The key here is notice that the the decomposition of $R-$modules

$$\text{Ind}_{R\rtimes I_N}^{R\rtimes G}(N\otimes L^\mu) = \bigoplus_{g_i\in G/I_N}(1\otimes g_i)N\otimes L^\mu \overset{g\rtimes g^{-1}(r)=rg}{\cong} \bigoplus_{g_i\in G/I_N} {}^{g_i}N\otimes L^\mu \tag{1}$$

shows $RG^{N,\mu}$ is a ss $R-$module as N is a simple $R-$module. Thus any $R\rtimes G$ submodule $M'\subseteq RG^{N,\mu}$ must also be a ss $R-$module. ss modules are direct sums of their isotypic components and Eq. (1) shows that all simple submodules of $RG^{N,\mu}$ are of the form ${}^gN$ and thus we see that as $R-$modules

$$\begin{aligned} M' &= \bigoplus_{g_i\in G/I_N} {}^{g_i}N\otimes \text{Hom}_R({}^{g_i}N, M') = \bigoplus_{g_i\in G/I_N} {}^{g_i}N\otimes\text{Hom}_R({}^{g_i}N, M'\cap g_i(N\otimes L^\mu)) \\ &\overset{G-\underline{submod}}{=\!=} \bigoplus_{g_i\in G/I_N} {}^{g_i}N\otimes\text{Hom}_R({}^{g_i}N, g_i(M'\cap(N\otimes L^\mu))\,) \cong \bigoplus_{g_i\in G/I_N} {}^{g_i}N\otimes\text{Hom}_R(N, M'\cap(N\otimes L^\mu)) \end{aligned}$$

Since M' is a $G-$submodule, $M'\cap(N\otimes L^\mu)$ is a I_N-submodule and thus we have the I_N-submodule

$$\text{Hom}_R(N, M'\cap(N\otimes L^\mu))\subset\text{Hom}_R(N, N\otimes L^\mu) = L^\mu$$

But L^μ is a simple I_N module and thus we have equality and thus $M' = RG^{N,\mu}$ as desired.
(b) Suppose M is a simple $R\rtimes G$ module and let N be any simple R submodule of M. Notice that $M = \sum_{g\in G} gN$ since the RHS is a $R\rtimes G$ submodule of M. We can then divide the sum as

$$M = \bigoplus_{g_i\in G/I_N} g_iL = \text{Ind}_{R\rtimes I_N}^{R\rtimes G} L \qquad \text{where } L = \sum_{h\in I_N} hN$$

L is a sum of isomorphic simple $R-$modules and thus is semisimple and thus is a direct sum of it's isotypic components so we have

$$L\cong N\otimes\text{Hom}_R(N, L)$$

One can then check that $\text{Hom}_R(N,L)$ is a simple $(I_N)_{\alpha_N^{-1}}$ module.
(c) Given $RG^{N_1,\mu} = RG^{N_2,\gamma}$ since N_1 is a $R-$ submodule of the LHS it follows that $N_1 = {}^gN_2$ for some $g\in G$. It then follows that $L^\mu = \text{Hom}_R(N_1, RG^{N_1,\mu})\cong\text{Hom}_R({}^gN_2, RG^{N_2,\gamma})\text{Hom}_R({}^gN_2, {}^gN_2\otimes L^\gamma) = L^\gamma$. ■

Remark. If G acts trivially on R then $R\rtimes\mathbb{C}[G] = R\otimes\mathbb{C}[G]$ and ${}^gN = N$ $\forall g\in G$ and so $I_N = G$ and thus the above recovers the fact that all simple $R\otimes\mathbb{C}[G]$ modules are of the form $N\otimes L$ where $N\in\text{Irr}(R), L\in\text{Irr}(G)$.

Lemma 1.5. *Let $e = (1/|G|)\sum_{g\in G} g\in R\rtimes G$.*

(a) We have a ring isomorphism

$$\theta : R^G \xrightarrow{\sim} e(R \rtimes G)e$$
$$s \mapsto se$$

(b) We have an isomorphism of $(R \rtimes G, R^G)$ bimodules

$$\psi : R \xrightarrow{\sim} (R \rtimes G)e$$
$$r \mapsto re$$

Proof. (a) Let $r \in R^G$, then expanding the first e we have

$$er = \frac{1}{|G|}\sum_{g\in G} g \rtimes r = \frac{1}{|G|}\sum_{g\in G} g(r)g \overset{r\in R^G}{=\!=} \frac{1}{|G|}\sum_{g\in G} rg = re \implies ere = re \tag{2}$$

And thus θ actually lands in $e(R \rtimes G)e$. Furthermore by Eq. (2), we see that for $r, s \in R^G$, $rese = rse$ so θ is a ring homomorphism. If $re = se$, then since $R \rtimes G$ is a free $R-$module with basis G, we have $r = s$. For surjectivity, like in Eq. (2) for $rh \in R \rtimes G$ we have the computation

$$e(rh)e = \frac{1}{|G|}\sum_{g\in G} g \rtimes rhe = \frac{1}{|G|}\sum_{g\in G} g(r)ghe \overset{ge=e\forall g\in G}{=\!=} \left(\frac{1}{|G|}\sum_{g\in G} g(r)\right) e \in R^G e$$

■

1.1. R^G simples

Lemma 1.6. *Let M, N be simple $(\mathbb{C}H)_\alpha-$modules and $e_H = \frac{1}{|H|}\sum_{h\in H} h$. Then*

$$\dim(e_H(M \otimes N^*)) = \begin{cases} 1 & \text{if } M \cong N \\ 0 & \text{otherwise} \end{cases}$$

Let N be a simple $R-$module. R^G acts on N on the left by restriction while $(\mathbb{C}I_N)_{\alpha_N}$ acts on N on the right by the isomorphisms $\phi_h : N \cong^h N$. Recall $\phi_h r = h(r)\phi_h$ and thus for $r \in R^G$, these two actions commute and we have a decomposition

$$N \cong \bigoplus_{\nu\in\widehat{(I_N)_{\alpha_N}}} N^\nu \otimes (L^\nu)^* \tag{3}$$

where $N^\nu \in R^G - \text{mod}$ and $L^\nu \in \mathbb{C}[I_N]_{\alpha_N} - \text{mod}$.

Theorem 2 *(a) If $N^\nu \neq 0$ then it's a simple R^G-module.*

(b) Every simple R^G-module is isomorphic to some N^ν and they are pairwise nonisomorphic.

Proof. It suffices to show $eRG^{N,\mu} \cong N^\nu$ and then (a) follows from Lemma 2.2 while (b) follows from Theorem 3 we know that all simple R^G. First note

$$\begin{aligned} eRG^{N,\mu} &:= e(R \rtimes G) \otimes_{R\rtimes H} (N \otimes L^\mu) \overset{eg=g}{=\!=} e \otimes_{R\rtimes H} (N \otimes L^\mu) \\ &= ee_H \otimes_{R\rtimes H} (N \otimes L^\mu) = e \otimes_{R\rtimes H} e_H(N \otimes L^\mu) \end{aligned}$$

Using the decomposition in Eq. (3) and Lemma 1.6 the result follows. ■

Corollary 1.7. *$N|_{R^G}$ is semisimple.*

2. Idempotent functor

Let S be any $\Bbbk$ algebra and $e \in S$ any idempotent and let $F_e : S - \text{mod} \to eSe - \text{mod}$ be defined as $F_e(V) = eV$.

Lemma 2.1. *F_e is exact.*

Proof. $eV \cong \text{Hom}_S(Se, V)$. ■

Lemma 2.2. *If $V \in S - \text{mod}$ is irreducible, then $F_e(V)$ is zero or irreducible.*

Proof. Let $W \subset eV$ be a eSe submodule. First note that $W = eSeW = eeSeW = eW$. Then $V = SW = SeW$ since V is an irreducible $S-$module. Thus

$$eV = e(SeW) = (eSe)W = W$$

Thus W has to be an irreducible eSe module if not zero. ■

Definition 2.3. *Let $V_{(e)}$ be the largest S submodule of V contained in $(1-e)V$.*

Lemma 2.4. *For $V \in S - \text{mod}$, the map $\pi_e : V \to V/V_{(e)}$ induces an isomorphism $F_e(\pi_e) : F_e(V) \to F_e(V/V_{(e)})$.*

Proof. F_e is exact and thus we have the exact sequence

$$0 \to F_e(V_{(e)}) \to F_e(V) \to F_e(V/V_{(e)}) \to 0$$

But $eV_{(e)} \subseteq e(1-e)V = 0$. ■

Let $G_e : eSe - \text{mod} \to S - \text{mod}$ be defined as $G_e(W) = S_e \otimes_{eSe} W$.

Lemma 2.5. *Given $W \in eSe - \text{mod}$, we have $F_e(G_e(W)) = e \otimes_{eSe} W \cong W$.*

Proof.

$$F_e(G_e(W)) = e(Se \otimes_{eSe}) = eSe \otimes_{eSe} W = e \otimes_{eSe} W \cong W$$

■

However G_e doesn't always take simples to simples and thus we still don't know if the map $F_e : S - \text{Irr} \to eSe - \text{Irr}$ is surjective or not. However, consider

Definition 2.6. *Let $G_e^* : eSe - \text{mod} \to S - \text{mod}$ be defined as $G_e^*(W) = G_e(W)/G_e(W)_{(e)}$.*

Proposition 2.7. *If $W \in eSe - \text{mod}$ is simple, then $G_e^*(W)$ is simple.*

Proof. It suffices to show that $G_e(W)_{(e)}$ is the unique maximal $S-$submodule of $G_e(W)$. Let $V = G_e(W)$. Then by the previous two lemmas we have

$$F_e(V/V_{(e)}) \cong F_e(V) = F_e(G_e(W)) \cong W$$

Thus $V/V_{(e)} \neq 0$ so $V_{(e)}$ is a proper submodule of V. Suppose V' is another proper $S-$ submodule of V. We claim $F_e(V') = eV' = 0$ and thus $V' \subseteq V_{(e)}$ showing $V_{(e)}$ is the unique maximal submodule. If not, then $F_e(V')$ is an eSe submodule of $eV = e \otimes_{eSe} W \cong W$ by Lemma 2.5 and thus equals $e \otimes_{eSe} W$ by simplicity. Therefore

$$V' \supseteq SeV' = Se \otimes_{eSe} W =: G_e(W) = V$$

which contradicts the fact that V' is a proper submodule. ■

Theorem 3
Suppose $\{V_\lambda \mid \lambda \in \Lambda\}$ is a full set of irreducible $S-$modules, and let $\Lambda' = \{\lambda \in \Lambda \mid eV_\lambda \neq 0\}$ Then $\{eV_\lambda \mid \lambda \in \Lambda'\}$ is a full set of distinct irreducibles $eSe-$modules.

Proof. From Proposition 2.7 we now have a map $G_e^* : eSe-\text{Irr} \to S-\text{Irr}$. We claim that $F_e(G_e^*(W)) \cong W$ and thus $F_e : S - \text{Irr} \to eSe - \text{Irr}$ is surjective. Indeed we have

$$F_e(G_e^*(W)) \overset{\text{Lemma 2.4}}{=\!=\!=} F_e(G_e(W)) \overset{\text{Lemma 2.5}}{\cong} W$$

For distinctness we need to show injectivity of F_e on $\{V_\lambda \,|\, \lambda \in \Lambda'\}$. We claim $G_e^*(F_e(V)) = V$ for $V \in S - \text{Irr}$. We have an $S - \text{mod}$ map

$$\begin{aligned} \beta : G_e(F_e(V)) = S_e \otimes_{eSe} eV &\to V \\ s \otimes_{eSe} ev &\mapsto sev \end{aligned}$$

The image is $SeV = V$ since V is irreducible $S-$module. Thus the kernel of β must be a maximal proper submodule of $G_e(F_e(V)$. But $F_e(V)$ is irreducible by Lemma 2.2 and thus Proposition 2.7 shows that the kernel of β must be $G_e(F_e(V))_{(e)}$. ■

Example 1. The Schur functor $S(n,r) - \text{mod} \to \mathbb{C}[S_r] - \text{mod}$ from the Schur algebra $S(n,r)$ to symmetric groups is a classical example of the above. Consider the weight $\omega = (1, 1, \ldots, 1, 0, \ldots, 0)$ with r 1's.